Error Propagation

The analysis of uncertainties (errors) in measurements and calculations is essential in the physics laboratory. For example, suppose you measure the length of a long rod by making three measurement $x = x_{best} \pm \Delta x$, $y = y_{best} \pm \Delta y$, and $z = z_{best} \pm \Delta z$. Each of these measurements has its own uncertainty Δx, Δy, and Δz respectively. What is the uncertainty in the length of the rod $L = x + y + z$? When we add the measurements do the uncertainties Δx, Δy, Δz cancel, add, or remain the same? Likewise , suppose we measure the dimensions $b = b_{best} \pm \Delta b$, $h = h_{best} \pm \Delta h$, and $w = w_{best} \pm \Delta w$ of a block. Again, each of these measurements has its own uncertainty Δb, Δh, and Δw respectively. What is the uncertainty in the volume of the block $V = bhw$? Do the uncertainties add, cancel, or remain the same when we calculate the volume? In order for us to determine what happens to the uncertainty (error) in the length of the rod or volume of the block we must analyze how the error (uncertainty) propagates when we do the calculation. In error analysis we refer to this as **error propagation**.

There is an error propagation formula that is used for calculating uncertainties when adding or subtracting measurements with uncertainties and a different error propagation formula for calculating uncertainties when multiplying or dividing measurements with uncertainties. Let's first look at the formula for adding or subtracting measurements with uncertainties.

Adding or Subtracting Measurements with Uncertainties.

Suppose you make two measurements,

$$x = x_{best} \pm \Delta x$$
$$y = y_{best} \pm \Delta y$$

What is the uncertainty in the quantity $q = x + y$ or $q = x - y$?
To obtain the uncertainty we will find the lowest and highest probable value of $q = x + y$. Note that we would like to state q in the standard form of $q = q_{best} \pm \Delta q$ where $q_{best} = x_{best} + y_{best}$.

(highest probable value of $q = x + y$):

$$(x_{best}+ \Delta x) + (y_{best} + \Delta y) = (x_{best}+ y_{best}) + (\Delta x +\Delta y) = q_{best} + \Delta q$$

(lowest probable value of $q = x + y$):

$$(x_{best}- \Delta x) + (y_{best} - \Delta y) = (x_{best}+ y_{best}) - (\Delta x +\Delta y) = q_{best} - \Delta q$$

Thus, we that

$$\Delta q = \Delta x + \Delta y$$

is the uncertainty in $q = x + y$. A similar result applies if we needed to obtain the uncertainty in the difference $q = x - y$. If we had added or subtracted more than two measurements x, y, ..., z each with its own uncertainty Δx, Δy, ... , Δz respectively , the result would be

$$\Delta q = \Delta x + \Delta y + \ldots\ldots\ldots + \Delta z$$

Now, if the uncertainties Δx, Δy, ..., Δz are random and independent, the result is

$$\boxed{\Delta q = \sqrt{(\Delta x)^2 + (\Delta y)^2 + \ldots\ldots\ldots + (\Delta z)^2}}$$

Ex. x = 3.52 cm ± 0.05 cm
y = 2.35 cm ± 0.04 cm

Calculate q = x + y
We would like to state q in the standard form of $q = q_{best} \pm \Delta q$

$x_{best} = 3.52cm$, $\Delta x = 0.05cm$

$y_{best} = 2.35cm$, $\Delta y = 0.04cm$

$q_{best} = x_{best} + y_{best} = 3.52cm + 2.35cm = 5.87cm$

$$\Delta q = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(0.05)^2 + (0.04)^2} = 0.06cm$$

q = 5.87cm ± 0.06cm

<u>Multiplying or Dividing Measurements with Uncertainties</u>

Suppose you make two measurements,

$$x = x_{best} \pm \Delta x$$
$$y = y_{best} \pm \Delta y$$

What is the uncertainty in the quantity q = xy or q = x/y?
To obtain the uncertainty we will find the highest and lowest probable value of q = xy. The result will be the same if we consider q = x/y. Again we would like to state q in the standard form of $q = q_{best} \pm \Delta q$ where now $q_{best} = x_{best}\, y_{best}$.

<u>(highest probable value of q = xy):</u>

$$(x_{best} + \Delta x)(y_{best} + \Delta y) = x_{best}y_{best} + x_{best}\,\Delta y + \Delta x\, y_{best} + \Delta x\,\Delta y = q_{best} + \Delta q$$

$$= x_{best}y_{best} + (x_{best}\,\Delta y + \Delta x\, y_{best}) = q_{best} + \Delta q$$

<u>(lowest probable value of q = xy):</u>

$(x_{best} - \Delta x)(y_{best} - \Delta y) = x_{best}y_{best} - x_{best}\,\Delta y - \Delta x\, y_{best} + \Delta x\,\Delta y = q_{best} - \Delta q$

$= x_{best}y_{best} - (x_{best}\,\Delta y + \Delta x\, y_{best}) = q_{best} - \Delta q$

Since the uncertainties Δx and Δy are assumed to be small, then the product $\Delta x\,\Delta y \approx 0$. Thus, we see that $\Delta q = x_{best}\,\Delta y + \Delta x\, y_{best}$ in either case. Dividing by $x_{best}y_{best}$ gives

$$\frac{\Delta q}{x_{best}y_{best}} = \frac{x_{best}\Delta y}{x_{best}y_{best}} + \frac{y_{best}\Delta x}{x_{best}y_{best}}$$

$$\frac{\Delta q}{q_{best}} = \frac{\Delta y}{y_{best}} + \frac{\Delta x}{x_{best}}$$

Again, a similar result applies if we needed to obtain the uncertainty in the division of q = x/y. If we had multiplied or divided more than two measurements x, y, ..., z each with its own uncertainty Δx, Δy, ... , Δz respectively, the result would be

$$\frac{\Delta q}{q_{best}} = \frac{\Delta y}{y_{best}} + \frac{\Delta x}{x_{best}} + \ldots\ldots + \frac{\Delta z}{z_{best}}$$

Now, if the uncertainties Δx, Δy, ..., Δz are random and independent, the result is

$$\boxed{\frac{\Delta q}{q_{best}} = \sqrt{\left(\frac{\Delta y}{y_{best}}\right)^2 + \left(\frac{\Delta x}{x_{best}}\right)^2 + \ldots\ldots + \left(\frac{\Delta z}{z_{best}}\right)^2}}$$

Ex. x = 49.52cm ± 0.08cm
y = 189.53cm ± 0.05cm

Calculate q = xy
We would like to state q in the standard form of $q = q_{best} \pm \Delta q$

$x_{best} = 49.52\text{cm}$, $\Delta x = 0.08\text{cm}$

$y_{best} = 189.53\text{cm}$, $\Delta y = 0.05\text{cm}$

$q_{best} = x_{best}y_{best} = (49.52\text{cm})(189.53\text{cm}) = 9.38553 \times 10^3\ \text{cm}^2$

$$\frac{\Delta q}{q_{best}} = \sqrt{\left(\frac{\Delta x}{x_{best}}\right)^2 + \left(\frac{\Delta y}{y_{best}}\right)^2} = \sqrt{\left(\frac{0.08cm}{49.52cm}\right)^2 + \left(\frac{0.05cm}{189.53cm}\right)^2} = 1.63691E-3$$

$\Delta q = (1.63691 \times 10^{-3}) q_{best} = (1.63691 \times 10^{-3})\,(9.38553 \times 10^{3}\ cm^{2})$

$\Delta q = 15.3632 cm^{2} \approx 20\ cm^{2}$

$q = 9390\ cm^{2} \pm 20\ cm^{2}$

<u>Uncertainty for a Quantity Raised to a Power</u>

If a measurement *x* has uncertainty Δx, then the uncertainty in $q = x^{n}$, is given by the expression

$$\frac{\Delta q}{q_{best}} = |n| \frac{\Delta x}{x_{best}}$$

Ex. Let $q = x^{3}$ where x = 5.75cm ± 0.08 cm.
Calculate the uncertainty Δq in the quantity q.

We would like to state q in the standard form of $q = q_{best} \pm \Delta q$

$n = 3$
$\Delta x = 0.08 cm$
$x_{best} = 5.75 cm$
$q_{best} = x_{best}^{3} = 190.1 cm^{3}$

$$\frac{\Delta q}{q_{best}} = |n| \frac{\Delta x}{x_{best}} =$$

$$\frac{\Delta q}{190.1 cm^{3}} = (3)\left(\frac{0.08 cm}{5.75 cm}\right)$$

$$\Delta q = 7.93 cm^{3} \approx 7 cm^{3}$$

$$q = 190 cm^{3} \pm 7 cm^{3}$$